# Differential Geometry III: Riemannian Geometry



**11.1** Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold.

(a) For any smooth function $f : \mathcal{M} \to \mathbb{R}$, we will define the Hessian $Hess[f]$ to be the $(0,2)$-tensor

$$Hess[f] \doteq \nabla df.$$

Show that, in any local coordinate system,

$$Hess[f]_{ij} = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f.$$

Deduce that $Hess(f)$ is a symmetric tensor. Show also that, for any $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$,

$$Hess[f](X,X) = \frac{d^2}{dt^2}\big(f \circ \gamma(t)\big)\Big|_{t=0}, \quad \text{where } \gamma \text{ is the geodesic } \gamma(t) = \exp_p(tX).$$

(b) For $f : \mathcal{M} \to \mathbb{R}$, let $c \in \mathbb{R}$ be such that $S = f^{-1}(\{c\})$ is a smooth hypersurface of $\mathcal{M}$ and $df \neq 0$ on $S$. Show that the scalar second fundamental form $b(\cdot,\cdot)$ of $S$ with respect to the coorientation determined by $\text{grad} f = df^\sharp$ is given by

$$b(X,Y) = -\frac{Hess[f](X,Y)}{\|\text{grad} f\|} \quad \text{for all } X, Y \in \Gamma(\mathcal{M}, S).$$

**11.2** Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold.

(a) The Einstein tensor $G$ of $(\mathcal{M}, g)$ is the $(0,2)$-tensor defined by

$$G = Ric - \frac{1}{2} S g,$$

where $S$ is the scalar curvature of $g$. Show that $G$ is divergence free, i.e.

$$g^{ab} \nabla_a G_{bc} = 0.$$

(*Hint: You might want to use the second Bianchi identity.*) Deduce that if $(\mathcal{M}, g)$ satisfies

$$Ric = \Lambda g$$

for some smooth function $\Lambda : \mathcal{M} \to \mathbb{R}$ and $\dim \mathcal{M} \geqslant 3$, then $\Lambda = \text{const}$ on each connected component of $\mathcal{M}$ (*Hint: Show first that, in this case, $G = \Lambda' g$ for some different function $\Lambda'$*). A Riemannian manifold satisfying such a relation is called an *Einstein manifold.*

(b) Show that if $(\mathcal{M}, g)$ is a connected Einstein manifold of dimension $\dim \mathcal{M} = 3$, then $(\mathcal{M}, g)$ has constant sectional curvature. (*Hint: Exercise 9.1.c might be helpful.*)

**Remark.** According to the theory of general relativity, a *vacuum* region of our spacetime (i.e. where matter is absent) is modelled by a Lorentzian manifold $(\mathcal{M}, g)$ satisfying $G = \Lambda g$, where $\Lambda$ is known as the *cosmological constant.* The above results indicate that non-trivial vacuum spacetimes exist only when $\dim \mathcal{M} \geqslant 4$.

**11.3** Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold.

(a) A 2-dimensional surface $S \subset \mathcal{M}$ is called *ruled* if, for every $p \in \mathcal{M}$, there exists a curve $\gamma : (-\delta, \delta) \to \mathcal{M}$ with $\gamma(0) = p$, $\dot{\gamma}(0) \neq 0$ which is a geodesic of $(\mathcal{M}, g)$ and lies entirely inside $S$. Show that, in this case,

$$\bar{K}_p \leqslant K[T_pS] \quad \text{for all } p \in S,$$

where $\bar{K}_p$ is the sectional curvature of $S$ with respect to the induced metric $\bar{g}$, while $K[T_pS]$ is the sectional curvature of the plane $T_pS \subset T_p\mathcal{M}$ with respect to the ambient metric $g$. This is known as *Synge's inequality.*

(b) Let $q$ be a point in $\mathcal{M}$ and let $\Omega \subset T_q\mathcal{M}$ be a convex open neighborhood of 0 such that $\exp_q$ is a diffeomorphism when restricted on $\Omega$. Let $S \subset \mathcal{M}$ be the surface defined by $S = \exp_q(\Omega \cap V)$, where $V$ is a 2-dimensional subspace of $T_q\mathcal{M}$. Show that $S$ is a ruled surface. Moreover, show that at the point $q$:

$$\bar{K}_q = K[T_qS].$$

**11.4** (a) Let $S \subset (R^3, g_E)$ be a smooth surface which is contained inside the ball

$$B_R = \left\{x \in \mathbb{R}^3 : \|x\| \leqslant R\right\}$$

and such that there exists a point $z \in S$ with $z \in \partial B_R$ (i.e. $\|z\| = R$). Deduce that $S$ and $S_R = \partial B_R$ have the same tangent plane at $z$. Show that the sectional curvature $K$ of $S$ satisfies at the point $z$

$$K_z \geqslant \frac{1}{R^2}.$$

*Hint: It might be useful to compare the sectional curvatures of $S$ and $S_R$ at $z$ by expressing both surfaces locally as graphs of functions defined over their common tangent plane $T_zS$ and use Exercise 9.1.*

(*b) A surface $S \subset \mathbb{R}^3$ is called *minimal* if it has vanishing mean curvature $H$ (such a surface is a stationary point of the total surface functional $\mathcal{A}[S] = \int_S d\bar{g}$, hence the name). Show that a minimal surface satisfies $K \leqslant 0$. Deduce that there is no compact minimal surface in $\mathbb{R}^3$. (*Hint: For a compact minimal surface $S$, start from a sphere completely surrounding $S$ and decrease its radius until you end up with a sphere both containing $S$ and touching $S$ at a point $z$.*)

* **11.5** Let $\gamma : [0, 1] \to \mathcal{M}$ be a geodesic of $(\mathcal{M}, g)$. Assume that there exist points $0 < a < b < 1$ and a vector field $Z$ along $\gamma$ with $Z \perp \dot{\gamma}$ satisfying the Jacobi equation

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Z - R(\dot{\gamma}, Z)\dot{\gamma} = 0$$

and such that

$$Z(a) = Z(b) = 0$$

with $Z$ not identically 0 on $[a, b]$. Show that $\gamma$ cannot be length minimizing among all curves connecting $\gamma(0)$ to $\gamma(1)$. *(Hint: You have to construct a variation $\phi_s$ of $\phi_0 = \gamma$ fixing the endpoints of $\gamma$ such that $\frac{d^2}{ds^2}(\ell(\phi_s))|_{s=0} < 0$. To this end, consider first the variation determined by a variation vector field which is equal to $Z$ in $[a, b]$ and 0 otherwise, and then consider small perturbations of this vector field around $t = a, b$ .)*